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Nonlocal interactions and quantum dynamics

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Abstract. The problem of describing the dynamics of quantum systems generated by a nonlocal-in-time interaction is considered. It is shown that the use of the Feynman approach to quantum theory in combination with the canonical approach allows one to extend quantum dynamics to describe the time evolution in the case of such interactions. In this way, using only the current concepts of quantum theory, a generalized equation of motion for state vectors is derived. In the case where the fundamental interaction generating the dynamics in a system is local in time, this equation is equivalent to the Schrödinger equation. Explicit examples are given for an exactly solvable model. The proposed formalism is shown to provide a new insight into the problem of the description of nonlocal interactions in quantum field theory. It is shown that such a property of the equation of motion, such as nonlocality in time, may be important for describing hadron–hadron interactions at low and intermediate energies.

1. Introduction

Various physical applications of quantum mechanics and quantum field theory require solutions of evolution problems. In standard canonical quantum theory, it is postulated that quantum dynamics is governed by the Schrödinger equation. However, as is well known, in quantum electrodynamics (QED) the ultraviolet divergences can be removed from the S matrix, but cannot be removed from quantities characterizing the time evolution of quantum systems, since regularization of the scattering matrix leads to a situation in which divergent terms automatically appear in the Schrödinger equation [1]. For this reason this equation is only of formal importance to quantum field theory. This leads, in particular, to difficulties in finding a consistent QED description of natural broadening of spectral line profiles in an atomic system [2]. Since locality has been argued to be the main cause of infinities in quantum field theory, it seems natural to resolve this problem by introducing a nonlocal form factor into the interaction Hamiltonian density. However, as is well known, such an introduction of a nonlocal form factor results in a loss of covariance. The reason for this is quite obvious. The Schrödinger equation is local in time, and the interaction Hamiltonian describes an instantaneous interaction. In nonrelativistic quantum mechanics processes of instantaneous interaction may be nonlocal in space. But in relativistic quantum theory a local-in-time process must also be local in space. Thus, for the introduction of nonlocality in the theory to be intrinsically consistent, one has to find a way of solving the evolution problem in the case when the dynamics in a system is generated by a nonlocal-in-time interaction. The solution of this problem may also be important for describing the low-energy hadron–hadron interaction. Indeed, nonlocality of interaction leads to the energy dependence of effective interaction operators. In recent years the possibility of using energy-dependent potentials to describe hadron–hadron interaction at low

and intermediate energies has been widely discussed [3–10]. Interest in studying potentials of this type is provoked by the still existing discrepancy between theory and experiment. For example, most ‘realistic’ nucleon–nucleon (NN) potentials are not sufficiently strong to reproduce the observed ${}^3\text{H}$ and ${}^3\text{He}$ binding energies (see, for example, [11]). Moreover, the energy dependence of the effective operators of hadron–hadron interaction is associated with the quark degrees of freedom, which are not included explicitly in the description of low-energy hadron interaction, and is predicted, for example, by quark models [3, 10]. However, the dynamics of such systems cannot be described in the framework of the Hamiltonian formalism. Indeed, the energy dependence of the interaction Hamiltonian means that the total Hamiltonian also depends on energy, i.e. on its spectral parameter. Such an ‘operator’ is not an operator in a rigorous sense. Hence the energy-dependent interaction operator cannot be interpreted as an interaction Hamiltonian.

Now let us turn to the Feynman formulation of quantum theory [12, 13]. The main idea of this formulation is that quantum dynamics can be described without resorting to the Schrödinger equation. Feynman’s theory starts with an analysis of the phenomenon of quantum interference which leads directly to the concept of the superposition of probability amplitudes. According to this concept, the probability amplitude of an event which can occur in several different ways is a sum of probability amplitudes for each of these ways [12]. The Feynman formulation also contains, as its essential idea, the concept of a probability amplitude associated with a completely specified motion or path in space-time, and it is postulated that this probability amplitude has a phase proportional to the action, computed classically, for the corresponding path. Using this postulate together with the above assumption concerning the calculation of probabilities in quantum mechanics leads to Feynman’s sum-over-paths formalism.

The theory of Feynman differs profoundly in its formulation from canonical quantum theory. These dissimilar approaches were proved to be equivalent and to complement one another in solving various problems in quantum physics. In this paper we show that these two approaches can be used in combination for describing the time evolution of quantum systems. In this way an equation of motion for state vectors is derived. Being equivalent to the Schrödinger equation in the case of local interaction, this equation makes it possible to solve the evolution problem in the case where the interaction generating the dynamics in a quantum system is nonlocal in time.

We start with the evolution equation $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, where $|\psi(t)\rangle$ is a state vector and $U(t, t_0)$ is the evolution operator. Then, using the basic assumption of the Feynman formulation, we represent the matrix elements of the evolution operator as a sum of contributions from all alternative ways of realization of the corresponding evolution process. The history of a quantum system is represented by some version of the time evolution of the system associated with completely specified instants of the beginning and end of the interaction in the system. In this way we obtain an expression for the matrix elements of the operator $U(t, t_0)$ in terms of probability amplitudes associated with such versions. An equation for these amplitudes is then derived from the requirement of unitarity for the evolution operator. It is shown that this equation can be regarded as an equation of motion. The concept of a generalized interaction operator is introduced. This operator is a generalization of the interaction Hamiltonian and generates the dynamics of a quantum system. In particular, the generalized interaction operator can be chosen so that the dynamics generated by this operator proves equivalent to the dynamics governed by the Schrödinger equation. At the same time, the equation of motion derived in this paper permits the generalization to the case where this equation manifests itself as a nonlocal-in-time dynamical equation. This point is illustrated in detail on an exactly solvable model. Finally, we discuss applications of the proposed formalism. This formalism is shown to provide a new insight into the problem of

the description of nonlocal interactions in quantum field theory. It is also shown that such a property of the proposed dynamical equation, as nonlocality in time may be important for describing hadron–hadron interactions at low and intermediate energies.

2. Basic assumptions of quantum theory

We assume the following properties for the states of quantum systems.

- (i) The physical state of a system is represented by a vector (properly by a ray) of a Hilbert space.
- (ii) An observable A is represented by a Hermitian hypermaximal operator α . The eigenvalues a_r of α give the possible values of A . An eigenvector $|\varphi_r^{(s)}\rangle$ corresponding to the eigenvalue a_r represents a state in which A has the value a_r . If the system is in the state $|\psi\rangle$, the probability P_r of finding the value a_r for A , when a measurement is performed, is given by

$$P_r = \langle\psi|P_{V_r}|\psi\rangle = \sum_s |\langle\varphi_r^{(s)}|\psi\rangle|^2$$

where P_{V_r} is the projection operator on the eigenmanifold V_r corresponding to a_r , and the sum \sum_s is taken over a complete orthonormal set $|\varphi_r^{(s)}\rangle$ ($s = 1, 2, \dots$) of V_r . The state of the system immediately after the observation is described by the vector $P_{V_r}|\psi\rangle$.

These assumptions are the main assumptions on which quantum theory is founded. In the canonical formalism these postulates are used together with the assumption that the time evolution of a state vector is governed by the Schrödinger equation. In our study we do not use this assumption. Instead, the following postulate is used.

- (iii) The probability of an event is the absolute square of a complex number called the probability amplitude. The joint probability amplitude of a time-ordered sequence of events is the product of the separate probability amplitudes of each of these events. The probability amplitude of an event which can happen in several different ways is a sum of the probability amplitudes for each of these ways.

The statements of the assumption (iii) express the well known law for the quantum mechanical probabilities. Within the canonical formalism this law is derived as one of the consequences of the theory. However, in the Feynman formulation of quantum theory this law is directly derived starting from the analysis of the phenomenon of quantum interference, and is used as a basic postulate of the theory. We also use the following assumption.

- (iv) Under space-time translations $x \rightarrow x+a$, the eigenstates $|n\rangle$ of a system of non-interacting particles corresponding to the total momentum of the particles $P = \sum_i p_i$ (p_i is the momentum of the i th particle) transforms as follows:

$$|n\rangle \xrightarrow{x \rightarrow x+a} \exp(-iP_n a)|n\rangle$$

where $E_n = \sum_i \sqrt{p_i^2 + m_i^2}$ (m_i are the masses of the particles), $P_n a \equiv P_n^\mu a_\mu$, and a is an arbitrary four-vector of displacement, and n stands for the entire set of discrete and continuous variables that characterize the system in full.

Here and below, we use units where $\hbar = 1$. The statement of the assumption (iv) expresses the well known law of transformation of vectors describing states of non-interacting particles under space-time translations. This law is used, for example, as one of the starting points in constructing the axiomatic approaches to quantum field theory (see, for example, [14, 15]).

The assumptions (i)–(iv) represent the current concepts of quantum theory. In this paper we suggest a new way of using these concepts. In this way an equation of motion more general than the Schrödinger equation is derived as a consequence of the assumptions (i)–(iv) and the requirement of conservation of probabilities.

3. Time-evolution operator

Let H_0 be the free Hamiltonian, i.e. the operator of the total energy of a system of particles travelling freely without interaction or external disturbance. The vectors $|n\rangle$ are the eigenvectors of this operator: $H_0|n\rangle = E_n|n\rangle$. As is well known, from assumption (ii) it follows that eigenvectors of any observable form a complete set of basis vectors in the Hilbert space. However, such continuum state vectors as the eigenvectors $|n\rangle$ of the free Hamiltonian H_0 do not belong to the Hilbert space of the states. Following the Dirac formalism we will assume $|n\rangle$ to be generalized basis vectors orthonormalized in a continuum (by using delta functions) in terms of which any vector of the Hilbert space can be expanded:

$$|\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle. \quad (1)$$

From this and assumption (iv) it follows that if there is no interaction in the system then any state vector has the following time dependence:

$$|\psi(t)\rangle = \exp(-iH_0t)|\psi(0)\rangle. \quad (2)$$

Below, we employ the interaction representation in which the state vectors relate to the state vectors in the Schrödinger representation as follows: $|\psi(t)\rangle_I = \exp(iH_0t)|\psi(t)\rangle_S$. From (2) it follows that in the interaction representation the state vectors of non-interacting particles are independent of time. Let us consider the probability amplitude $\langle\psi_2|U(t, t_0)|\psi_1\rangle$ of finding, for a measurement at time t , the quantum system in the state $|\psi_2\rangle$ if at time t_0 it was in the state $|\psi_1\rangle$. As follows from assumption (ii), this probability is the absolute square of the probability amplitude which is given by the scalar product $\langle\psi_2|\psi(t)\rangle$. Here $|\psi(t)\rangle = U(t, t_0)|\psi_1\rangle$ is the state at time t to which the system evolves from the state $|\psi_1\rangle$ at time t_0 if it is not disturbed by measurements; accordingly, $U(t, t_0)$ is the operator describing such an evolution. In the canonical approach to quantum theory $U(t, t_0)$ is postulated to be a unitary operator

$$U^\dagger(t_2, t_1)U(t_2, t_1) = U(t_2, t_1)U^\dagger(t_2, t_1) = \mathbf{1} \quad (3)$$

with the group property

$$U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0) \quad U(t_0, t_0) = \mathbf{1}. \quad (4)$$

In the case of an isolated system (only systems of this type are considered in this paper), the evolution operator in the Schrödinger picture $U_s(t_2, t_1) \equiv \exp(-iH_0t_2)U(t_2, t_1)\exp(iH_0t_1)$ depends only on the difference $(t_2 - t_1)$ only, so that the operators $V(t) \equiv U_s(t, 0)$ constitute a one-parameter group of unitary operators, with the group property

$$V(t_1 + t_2) = V(t_1)V(t_2) \quad V(0) = \mathbf{1}. \quad (5)$$

If these operators are assumed to be strongly continuous, i.e. if

$$\lim_{t_2 \rightarrow t_1} \|V(t_2)|\psi\rangle - V(t_1)|\psi\rangle\| = 0 \quad (6)$$

then from Stone's theorem it follows [16] that this one-parameter group has a self-adjoint infinitesimal generator H :

$$V(t) = \exp(-iHt) \quad \text{id}/\text{dt} V(t) = HV(t).$$

Identifying H with the total Hamiltonian as usual, we get the time-dependent Schrödinger equation: $i\frac{d|\psi_s(t)\rangle}{dt} = H|\psi_s(t)\rangle$, where $|\psi_s(t)\rangle = V(t)|\psi_s(t=0)\rangle$. However, condition (6) seems to be too strong. From the physical point of view, it is enough to require that

$$\langle\psi_2|V(t_2)|\psi_1\rangle \xrightarrow{t_2 \rightarrow t_1} \langle\psi_2|V(t_1)|\psi_1\rangle \quad (7)$$

for any physically realizable states $|\psi_1\rangle$ and $|\psi_2\rangle$ [14]. Note, in this connection, that there are normalized vectors in the Hilbert space that represent the states for which the energy of a system is infinite. Such states cannot be considered as physically realizable [14], and hence the corresponding matrix elements of the evolution operator need not be continuous. For this reason, in this paper we do not restrict ourselves to the strongly continuous evolution operators. We only assume that the evolution operator satisfies the condition (7). This means that we do not use the assumption that the time evolution of a state vector is governed by the Schrödinger equation as the basic dynamical postulate.

According to assumption (iii), the probability amplitude of an event which can happen in several different ways is a sum of contributions from each alternative way. In particular, the amplitude $\langle\psi_2|U(t, t_0)|\psi_1\rangle$ can be represented as a sum of contributions from all alternative ways of realization of the corresponding evolution process. Dividing these alternatives in different classes, we can then analyse such a probability amplitude in different ways [13]. For example, subprocesses with definite instants of the beginning and end of the interaction in the system can be considered as such alternatives. Let $\langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle$ be the probability amplitude that if at time t_0 the system was in the state $|n_1\rangle$, then the interaction in the system will begin at time t_1 and end at time t_2 , and at time t the system will be in the state $|n_2\rangle$. According to postulate (iii), the amplitudes $\langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle$ determine the contributions from the above alternatives, and $\langle n_2|U(t, t_0)|n_1\rangle$ can be represented as a sum (more precisely as an integral) of these contributions

$$\langle n_2|U(t, t_0)|n_1\rangle = \varphi(n_2, n_1, t, t_0) + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle \quad (8)$$

where $\varphi(n_2, n_1, t, t_0)$ is the probability amplitude that if at time t_0 the system was in the state $|n_1\rangle$, then the particles of the system will not interact in the time interval (t_0, t) , and at time t the system will be in the state $|n_2\rangle$. Thus, the first term on the right-hand side of (8) is the contribution from the alternative subprocess in the case of which the particles of the system do not interact. From postulate (iii), it follows that the probability amplitude $\langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle$ is expressible as a product

$$\langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle = \sum_{n'} \sum_n \varphi(n_2, n', t, t_2) \langle n'|\tilde{S}(t_2, t_1)|n\rangle \varphi(n, n_1, t_1, t_0) \quad (9)$$

where $\langle n_2|\tilde{S}(t_2, t_1)|n_1\rangle$ is the probability amplitude that if at time t_1 the system was in the state $|n_1\rangle$, then the interaction in the system will begin at time t_1 and will end at time t_2 , and at this time the system will be in the state $|n_2\rangle$.

The evolution operator $U(t, t_0)$ has a natural decomposition

$$U(t, t_0) = \mathbf{1} + iR(t, t_0). \quad (10)$$

Here the unit operator represents the no-interaction part; its matrix elements are delta functions which make the final momenta the same as the initial momenta. The operator $R(t, t_0)$ represents the interaction part. From (8) and (10) it follows that $\varphi(n_2, n_1, t_2, t_1) = \langle n_2|n_1\rangle$, and hence $\langle n_2|U(t, t_0; t_2, t_1)|n_1\rangle = \langle n_2|\tilde{S}(t_2, t_1)|n_1\rangle$. Thus, equation (8) can be rewritten in the form

$$\langle n_2|U(t, t_0)|n_1\rangle = \langle n_2|n_1\rangle + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \langle n_2|\tilde{S}(t_2, t_1)|n_1\rangle \quad (11)$$

accordingly, the evolution operator $U(t, t_0)$ can be expressed in terms of the operators $\tilde{S}(t_2, t_1)$ whose matrix elements are the amplitudes $\langle n_2 | \tilde{S}(t_2, t_1) | n_1 \rangle$:

$$U(t, t_0) = \mathbf{1} + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \tilde{S}(t_2, t_1). \quad (12)$$

Equation (12) defines the evolution operator $U(t_2, t_1)$ only for $t_2 \geq t_1$. Using (3), the evolution operator $U(t_2, t_1)$ for $t_2 < t_1$ can be constructed as follows:

$$U(t_2, t_1) = U^{-1}(t_1, t_2) = U^+(t_1, t_2). \quad (13)$$

It should be noted that in general $\tilde{S}(t_2, t_1)$ may be an operator-valued distribution, since only $U(t, t_0)$ has to be an operator on the Hilbert space. Nevertheless, throughout this paper we use the term ‘operator’ for $\tilde{S}(t_2, t_1)$.

To clarify the role which the operator $\tilde{S}(t_2, t_1)$ plays in the proposed formalism, note the following. The Feynman formulation is based on the assumption that the history of a system can be represented by some path in space-time. From postulate (iii) it then follows that the probability amplitudes of any event is a sum of the probability amplitudes that a particle has a completely specified path in space-time. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of \hbar) for the path in question. In the proposed formalism the history of a system is represented by the version of the time evolution of the system associated with completely specified instants of the beginning and end of the interaction in the system. Such a description of the history of a system is more general and requires no supplementary postulates like the above assumptions of the Feynman formulation. On the other hand, the probability amplitudes $\langle \psi_2 | \tilde{S}(t_2, t_1) | \psi_1 \rangle$ in terms of which we describe quantum dynamics, are used in the spirit of Feynman’s theory: the probability amplitude of any event is represented as a sum of these amplitudes. Below we show that the requirement of unitarity for the evolution operator given by (12) leads to an equation for the operator $\tilde{S}(t_2, t_1)$ which can be regarded as an equation of motion.

As we have noted, $\langle \psi(t_2) | \tilde{S}(t_2, t_1) | \psi(t_1) \rangle$ is the probability amplitude that if at time t_1 the system was in the state $|\psi_1\rangle$, then the interaction in the system will begin at time t_1 and end at time t_2 , and at this time the system will be found in the state $|\psi(t_2)\rangle$. Here this probability amplitude is represented by the matrix element $\langle \psi(t_2) | \tilde{S}(t_2, t_1) | \psi(t_1) \rangle$ in the interaction picture. However, the same probability amplitude can be represented by the matrix element $\langle \psi_s(t_2) | \tilde{S}_s(t_2, t_1) | \psi_s(t_1) \rangle$ in the Schrödinger picture, where $\tilde{S}_s(t_2, t_1)$ is the operator describing the transformation of a state at time t_1 into the state at time t_2 caused by the interaction in the system that begins at time t_1 and ends at time t_2 . Being an operator in the Schrödinger picture, $\tilde{S}_s(t_2, t_1)$ depends on the difference $(t_2 - t_1)$ only: $\tilde{S}_s(t_2, t_1) \equiv \tilde{T}(t_2 - t_1)$. Since $\langle \psi_s(t_2) | \tilde{S}(t_2, t_1) | \psi_s(t_1) \rangle$ and $\langle \psi_s(t_2) | \tilde{T}(t_2 - t_1) | \psi_s(t_1) \rangle$ represent the same probability amplitude, we have

$$\langle \psi(t_2) | \tilde{S}(t_2, t_1) | \psi_s(t_1) \rangle = \langle \psi_s(t_2) | \tilde{T}(t_2 - t_1) | \psi_s(t_1) \rangle. \quad (14)$$

Taking into account the relation between the states in the Schrödinger and interaction pictures given by $|\psi(t)\rangle = \exp(iH_0 t) |\psi_s(t)\rangle$, from (14), we get

$$\tilde{S}(t_2, t_1) = \exp(iH_0 t_2) \tilde{T}(t_2 - t_1) \exp(-iH_0 t_1). \quad (15)$$

Let us now consider the scattering matrix. Letting $t_0 \rightarrow -\infty$ and $t \rightarrow \infty$ in (12), we get the S matrix in the following form:

$$\langle n_2 | S | n_1 \rangle = \langle n_2 | n_1 \rangle + \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \langle n_2 | \tilde{S}(t_2, t_1) | n_1 \rangle. \quad (16)$$

By using (15), this expression can be rewritten in the form

$$\langle n_2 | S | n_1 \rangle = \langle n_2 | n_1 \rangle - 2\pi i \delta(E_{n_2} - E_{n_1}) \langle n_2 | T(E_{n_1}) | n_1 \rangle \quad (17)$$

where the operator $T(z)$ is defined by

$$\langle n_2 | T(z) | n_1 \rangle = i \int_{-\infty}^{\infty} d\tau \exp(iz\tau) \langle n_2 | F(\tau) | n_1 \rangle \quad (18)$$

with

$$\langle n_2 | F(\tau) | n_1 \rangle = \langle n_2 | \tilde{T}(\tau) | n_1 \rangle \quad \tau \geq 0 \quad (19)$$

and $\langle n_2 | F(\tau) | n_1 \rangle = 0$ for $\tau < 0$. The fact that $F(\tau) = 0$ for $\tau < 0$ can be considered as an expression of causality, since τ is assumed to be the duration time of interaction that has to be positive. A remarkable consequence of this fact is that $\langle n_2 | T(z) | n_1 \rangle$ is analytic in the upper half of the complex z plane.

4. Unitarity condition and its consequences

Let us consider the unitarity condition that expresses the principle of conservation of probabilities. Substituting (10) into the unitarity condition (3) gives

$$\langle n_2 | R(t, t_0) | n_1 \rangle - \langle n_2 | R^+(t, t_0) | n_1 \rangle = i \sum_n \langle n_2 | R^+(t, t_0) | n \rangle \langle n | R(t, t_0) | n_1 \rangle \quad (20)$$

$$\langle n_2 | R(t, t_0) | n_1 \rangle - \langle n_2 | R^+(t, t_0) | n_1 \rangle = i \sum_n \langle n_2 | R(t, t_0) | n \rangle \langle n | R^+(t, t_0) | n_1 \rangle. \quad (21)$$

From (10), (12) and (15) it follows that the operator $R(t, t_0)$ can be written as

$$R(t, t_0) = -i \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \exp(iH_0 t_2) \tilde{T}(t_2 - t_1) \exp(-iH_0 t_1). \quad (22)$$

Taking into account that, according to (18), the operator $\tilde{T}(\tau)$ can be represented in the form

$$\tilde{T}(\tau) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-iz\tau) T(z) \quad (23)$$

where $z = x + iy$, x and y are real, and $y > 0$, we get

$$\begin{aligned} \langle n_2 | R(t, t_0) | n_1 \rangle &= -\frac{1}{2\pi} \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \int_{-\infty}^{\infty} dx \\ &\times \exp(iE_{n_2} t_2) \exp[-iz(t_2 - t_1)] \exp(-iE_{n_1} t_1) \langle n_2 | T(z) | n_1 \rangle. \end{aligned} \quad (24)$$

Let us assume that $\langle n_2 | T(z) | n_1 \rangle$ satisfies the condition

$$\lim_{|z| \rightarrow \infty} \frac{\langle n_2 | T(z) | n_1 \rangle}{z} = 0. \quad (25)$$

Note in this connection that the properties of $\langle n_2 | T(z) | n_1 \rangle$ at infinity depend on the behaviour of $\langle n_2 | \tilde{S}(t_2, t_1) | n_1 \rangle$ in the limit $t_2 \rightarrow t_1$. As shown below, in the proposed formalism this behaviour determines the dynamics in the system. Thus, by assuming that $\langle n_2 | T(z) | n_1 \rangle$ satisfies condition (25), we restrict ourselves to considering some class of the dynamical schemes. However, the important thing for us is that, as is shown below, the Hamiltonian dynamics belongs to this class. Taking into account condition (25), from (24), we get (see appendix A)

$$\langle n_2 | R(t, t_0) | n_1 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\exp[-i(z - E_{n_2})t] \exp[i(z - E_{n_1})t_0]}{(z - E_{n_2})(z - E_{n_1})} \langle n_2 | T(z) | n_1 \rangle. \quad (26)$$

Substituting (26) into (20) gives

$$\begin{aligned} & \int_{-\infty}^{\infty} dx_1 \frac{\exp[-i(z_1 - E_{n_2})t] \exp[i(z_1 - E_{n_1})t_0]}{(z_1 - E_{n_2})(z_1 - E_{n_1})} \langle n_2 | T(z_1) | n_1 \rangle \\ & - \int_{-\infty}^{\infty} dx_2 \frac{\exp[i(z_2^* - E_{n_1})t] \exp[-i(z_2^* - E_{n_2})t_0]}{(z_2^* - E_{n_2})(z_2^* - E_{n_1})} \langle n_2 | T^+(z_2) | n_1 \rangle \\ & = \langle n_2 | M(t, t_0) | n_1 \rangle \end{aligned} \quad (27)$$

where $z_1 = x_1 + iy$, $z_2 = x_2 + iy$, $y > 0$, and

$$\begin{aligned} \langle n_2 | M(t, t_0) | n_1 \rangle &= \frac{i}{2\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \\ & \times \frac{\exp[i(z_2^* - z_1)(t - t_0)] \exp[i(E_{n_2} - E_{n_1})t_0]}{(z_2^* - E_{n_2})(z_1 - E_{n_1})} \langle n_2 | B(z_2, z_1) | n_1 \rangle \end{aligned} \quad (28)$$

with

$$\langle n_2 | B(z_2, z_1) | n_1 \rangle = \sum_n \frac{\langle n_2 | T^+(z_2) | n \rangle \langle n | T(z_1) | n_1 \rangle}{(z_2^* - E_n)(z_1 - E_n)}. \quad (29)$$

Note that equation (27) derived in this way must be satisfied for $t \geq t_0$. However, taking into account the analytic properties of $\langle n_2 | T(z) | n_1 \rangle$, one can easily verify that both sides of (27) are equal to zero for $t < t_0$. Thus, equation (27) is valid for all t and t_0 . Represent the operator $B(z_1, z_2)$ in the form

$$B(z_1, z_2) = \frac{T(z_1) - T^+(z_2)}{z_2^* - z_1} + F(z_1, z_2). \quad (30)$$

Substituting (30) into (27) and taking into account the analytic properties of $\langle n_2 | T(z) | n_1 \rangle$, we then have

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \frac{\exp[i(z_2^* - z_1)(t - t_0)] \exp[i(E_{n_2} - E_{n_1})t_0]}{(z_2^* - E_{n_2})(z_1 - E_{n_1})} \langle n_2 | F(z_1, z_2) | n_1 \rangle = 0. \quad (31)$$

Since equation (31) must be satisfied for all t and t_0 , we get

$$\langle n_2 | F(z_1, z_2) | n_1 \rangle = 0. \quad (32)$$

From this it follows that

$$T(z_1) - T^+(z_2) = (z_2^* - z_1) \sum_n \frac{T^+(z_2) | n \rangle \langle n | T(z_1)}{(z_2^* - E_n)(z_1 - E_n)}. \quad (33)$$

Proceeding in an analogous way, from (21), we obtain

$$T(z_1) - T^+(z_2) = (z_2^* - z_1) \sum_n \frac{T(z_1) | n \rangle \langle n | T^+(z_2)}{(z_2^* - E_n)(z_1 - E_n)}. \quad (34)$$

It is easy to show then that the following equation results from (33) and (34):

$$T(z_1) - T(z_2) = (z_2 - z_1) \sum_n \frac{T(z_2) | n \rangle \langle n | T(z_1)}{(z_2 - E_n)(z_1 - E_n)}. \quad (35)$$

Thus, we have shown that for the evolution operator given by (12) to be unitary the operator $T(z)$ must satisfy equations (33)–(35). At the same time, as it shown in appendix B, the evolution operator given by (12) satisfies the composition law (4), provided equation (35) is valid. Note that equation (35) can be derived from the composition law (4) and the representation (12) in the same way in which (33) has been derived from this representation and the unitarity condition (20). However, equations (33) and (34) are more general than equation (35). In

fact, (35) results from (33) and (34). But for (33) and (34) to result from (35) the additional assumption that $T^+(z) = T(z^*)$ is required.

Equation (35) plays a key role in the proposed formalism. It has been derived as a consequence of such fundamental physical principles of quantum theory as the requirement of conservation of probabilities and the principle of the superposition of probability amplitudes. It should be noted that an equation having the same form as equation (35) has been derived for the scattering T matrix as a consequence of the above principles within the approach to the scattering theory developed in [17–20]. That equation was derived as an off-shell generalization of the unitarity condition for the scattering S matrix, and may be regarded as a particular case of equation (35).

5. Equation of motion for state vectors

In the previous section we derived equations (33)–(35) for the operator $T(z)$. It now becomes necessary for us to obtain an equation directly for the operator $\tilde{S}(t_2, t_1)$ which is of central importance in our formalism. It can be shown (see appendix C) that for equation (35) to be valid for any z_1 and z_2 the operator must satisfy the following equations:

$$(1 - \exp[ia(t_2 - t_1)])\tilde{S}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 \exp[ia(t_2 - t_4)](1 - \exp[ia(t_4 - t_3)])\tilde{S}(t_2, t_4)\tilde{S}(t_3, t_1) \quad (36)$$

$$(t_2 - t_1)\tilde{S}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3)\tilde{S}(t_2, t_4)\tilde{S}(t_3, t_1) \quad (37)$$

where $a = z_2 - z_1$. Substituting (15) into (37), we then obtain the corresponding equation for the operator $\tilde{T}(\tau)$

$$\tau \tilde{T}(\tau) = \int_0^\tau d\tau_2 \int_0^{\tau-\tau_2} d\tau_1 (\tau - \tau_1 - \tau_2) \tilde{T}(\tau_2) \exp[-i(\tau - \tau_1 - \tau_2)H_0] \tilde{T}(\tau_1). \quad (38)$$

It should be noted that (38) coincides, in form, with the equation for the operator $\tilde{T}(\tau)$ derived within scattering theory [21]. However, (38) was initially considered as an equation allowing one to ascertain which boundary conditions for equation (35) are admissible. In [22] it was shown that this equation enables one to construct a new class models in nonrelativistic quantum scattering theory, which were shown to open new possibilities for describing hadron–hadron interactions at low energies.

As we show below, equation (37) allows one to obtain the amplitudes $\langle n_2 | \tilde{S}(t_2, t_1) | n_1 \rangle$ for any t_1 and t_2 if the amplitudes $\langle n_2 | \tilde{S}(t'_2, t'_1) | n_1 \rangle$ corresponding to infinitesimal duration times $\tau = t'_2 - t'_1$ of interaction are known. It is natural to assume that most of the contribution to the evolution operator in the limit $t_2 \rightarrow t_1$ comes from the processes associated with an fundamental interaction in the system under study. Denoting this contribution by $H_{int}(t_2, t_1)$, the operator $\tilde{S}(t_2, t_1)$ can be represented in the form

$$\tilde{S}(t_2, t_1) = H_{int}(t_2, t_1) + \tilde{S}_1(t_2, t_1) \quad (39)$$

where $\tilde{S}_1(t_2, t_1)$ is the part of the operator $\tilde{S}(t_2, t_1)$ which in the limit $t_2 \rightarrow t_1$ gives the negligibly small contribution to the evolution operator in comparison with $H_{int}(t_2, t_1)$. We assume that the operator $H_{int}(t_2, t_1)$ contains all the dynamical information that is needed to construct the evolution operator $U(t_2, t_1)$. Thus we assume that the operator $H_{int}(t_2, t_1)$ plays the role which the interaction Hamiltonian plays in the ordinary formulation of quantum theory: it generates dynamics in a system. This operator can be regarded as a generalization of the

interaction Hamiltonian, and we call it the generalized interaction operator. Obviously, the operator $H_{int}(t_2, t_1)$ must satisfy (37) in the limit $t_2 \rightarrow t_1$

$$F_1(t_2, t_1) \xrightarrow{t_2 \rightarrow t_1} 0 \quad (40)$$

where

$$F_1(t_2, t_1) = -(t_2 - t_1)H_{int}(t_2, t_1) + \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3)H_{int}(t_2, t_4)H_{int}(t_3, t_1).$$

According to (15), the operator $H_{int}(t_2, t_1)$ can be represented in the form

$$H_{int}(t_2, t_1) = \exp(iH_0 t_2) H_{int}^{(s)}(t_2 - t_1) \exp(-iH_0 t_1) \quad (41)$$

where $H_{int}^{(s)}(t_2 - t_1)$ is the generalized interaction operator in the Schrödinger picture. If $H_{int}(t_2, t_1)$ is specified, equation (37) allows one to find the operator $\tilde{S}(t_2, t_1)$. Formula (12) can then be used to construct the evolution operator $U(t, t_0)$ and accordingly the state vector

$$|\psi(t)\rangle = |\psi(t_0)\rangle + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \tilde{S}(t_2, t_1) |\psi(t_0)\rangle \quad (42)$$

at any time t . Thus (37) can be regarded as an equation of motion for states of a quantum system.

From the mathematical point of view the requirement that $H_{int}(t_2, t_1)$ contains all the dynamic information that is needed for constructing $U(t_2, t_1)$ means that the operator $H_{int}(t_2, t_1)$ must have such a form that equation (37) has a unique solution having the following behaviour near the point $t_2 = t_1$:

$$\tilde{S}(t_2, t_1) \xrightarrow{t_2 \rightarrow t_1} H_{int}(t_2, t_1) + o(\tau^\epsilon) \quad (43)$$

where $\tau = t_2 - t_1$ and the value of ϵ depends on the form of the operator $H_{int}(t_2, t_1)$. In order to clarify this point, note that equation (38) is equivalent to the following equation for the operator $T(z)$ given by (18):

$$\frac{dT(z)}{dz} = T(z)G^{(2)}(z)T(z) \quad (44)$$

where

$$G^{(2)}(z) = - \sum_n \frac{|n\rangle\langle n|}{(z - E_n)^2}.$$

The correspondence between equation (38) and the differential equation (44) can be easily stated in the same way in which we stated the correspondence between equations (35) and (36). Thus, instead of solving (37) or (38), one can solve equation (44) for the operator $T(z)$. The operator $\tilde{T}(\tau)$ and correspondingly the operator $\tilde{S}(t_2, t_1)$ can then be obtained by using (23). At the same time, according to (26), the operator $T(z)$ can be used directly for constructing the evolution operator. According to (18), (39) and (41), the operator $T(z)$ has the following asymptotic behaviour for $|z| \rightarrow \infty$:

$$T(z) \xrightarrow{|z| \rightarrow \infty} B(z) + o(|z|^{-\beta}) \quad (45)$$

where

$$B(z) = i \int_0^\infty d\tau \exp(iz\tau) H_{int}^{(s)}(\tau) \quad (46)$$

and $\beta = 1 + \epsilon$. From (44) and (45) it follows that the operator $B(z)$ must satisfy the following asymptotic condition:

$$\frac{dB(z)}{dz} \xrightarrow{|z| \rightarrow \infty} B(z)G^{(2)}(z)B(z) + o(|z|^{-\beta}). \quad (47)$$

The above requirements, which the operator $H_{int}^{(s)}(\tau)$ has to meet, mean that $B(z)$ must be so close to the solution of equation (44) in the limit $|z| \rightarrow \infty$ that this differential equation has a unique solution having the asymptotic behaviour (45). The operator $B(z)$ represents the contribution which $H_{int}^{(s)}(\tau)$ gives to the operator $T(z)$, and we call it the effective interaction operator.

Let us now show that the Schrödinger equation results from equation (37) and hence the dynamics of a quantum system is equivalent to the Hamiltonian dynamics in the case when the generalized interaction operator is of the form

$$H_{int}^{(s)}(\tau) = -2i\delta(\tau)H_I \quad (48)$$

H_I being a self-adjoint operator. The delta function $\delta(\tau)$ in (48) emphasizes the fact that in this case, the fundamental interaction is instantaneous. As it is shown in appendix C, from (37) it follows that $\tilde{S}(t_2, t_1)$ must also satisfy equation (36) for any a . Letting $a \rightarrow i\infty$ in (36) and taking into account (48), we get the following equation:

$$\langle n_2 | \tilde{S}(t_2, t_1) | n_1 \rangle = -2i\delta(t_2 - t_1) \langle n_2 | H_I(t_1) | n_1 \rangle - i \sum_n \int_{t_1}^{t_2} dt_3 \langle n_2 | H_I(t_2) | n \rangle \langle n | \tilde{S}(t_3, t_1) | n_1 \rangle \quad (49)$$

where $H_I(t) = \exp(iH_0 t) H_I \exp(-iH_0 t)$. Solving this equation by expanding $\tilde{S}(t_2, t_1)$ in terms of $H_I(t)$, one can easily obtain

$$\begin{aligned} \tilde{S}(t, t_0) = & -2i\delta(t - t_0)H_I(t_0) \\ & + \sum_{n=2}^{\infty} (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-3}} dt_{n-2} H_I(t) H_I(t_1) \dots H_I(t_{n-2}) H_I(t_0). \end{aligned} \quad (50)$$

Thus, we have shown that for the case under study, equation (37) allows one to obtain $\tilde{S}(t_2, t_1)$ for any t_1 and t_2 starting from the contribution to the evolution operator coming from the process associated with the instantaneous interaction.

Inserting (50) into (12) yields

$$U(t, t_0) = \mathbf{1} + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \dots H_I(t_n). \quad (51)$$

This expression coincides, in form, with the Dyson expansion of the evolution operator. From this it follows that the operator $H_I(t)$ has to be identified with the interaction Hamiltonian. As is well known, the series (51) is convergent, provided the operator $H_I(t)$ is bounded. This means that the series (50) is also convergent in this case. On the other hand, substituting (49) into (12), we easily obtain the following equation for the evolution operator $U(t_2, t_1)$:

$$U(t_2, t_1) = \mathbf{1} - i \int_{t_1}^{t_2} dt H_I(t) U(t, t_1). \quad (52)$$

This equation is the integral form of the Schrödinger equation for the evolution operator

$$\frac{dU(t, t_0)}{dt} = -iH_I(t)U(t, t_0) \quad U(t_0, t_0) = \mathbf{1}. \quad (53)$$

Hence, in the case when $H_{int}^{(s)}(\tau)$ is of the form (48), from (37) it follows that $|\psi(t)\rangle$ given by (42) satisfies the Schrödinger equation

$$\frac{d|\psi(t)\rangle}{dt} = -iH_I(t)|\psi(t)\rangle.$$

Thus we have shown the equivalence of the dynamics governed by equation (37) to the Hamiltonian dynamics in the case when the generalized interaction operator $H_{int}^{(s)}(\tau)$ is of the form (48). At the same time, as we show below, equation (37) permits the generalization to the case when the operator $H_{int}^{(s)}(\tau)$ has no such a singularity as the delta function at the point $t_2 = t_1$. In this case the fundamental interaction generating the dynamics in a quantum system is nonlocal-in-time: the evolution operator is defined by $H_{int}^{(s)}(\tau)$ as a function of the time duration τ of the interaction.

It should be noted that the concept of nonlocal-in-time potentials was first introduced within the optical-potential model. This concept, for example, is used in the theory of time-dependent optical potentials (see, for example, [23, 24] and references therein). The optical potentials are introduced in the case when only state vectors belonging some subspace of the Hilbert space are included explicitly in the description of the time evolution of a quantum system. Such potentials which account for the coupling between this subspace and its complementary part of the Hilbert space are nonlocal in time and, thus, depends on the history of a dynamical system. The nonlocal form of the optical potentials is an expression of the loss of probability from the above subspace. However, the optical-potential model is one of the methods used for the description of the time evolution of quantum systems, the dynamics of which is generated by the local-in-time interaction being described by the interaction Hamiltonian. In the present formalism such a dynamics corresponds to the particular case when the generalized interaction operator is of the form (48). At the same time our formalism permits the generalization to the case when the fundamental interaction generating the dynamics in a quantum system is itself nonlocal in time and hence the nonlocality of the interaction does not lead to the loss of probability in the system.

6. Exactly solvable model

Let us consider the evolution problem for two nonrelativistic particles in the c.m.s. We denote the relative momentum by \mathbf{p} and the reduced mass by μ . Assume that the generalized interaction operator in the Schrödinger picture $H_{int}^{(s)}(\tau)$ has the form

$$\langle \mathbf{p}_2 | H_{int}^{(s)}(\tau) | \mathbf{p}_1 \rangle = \varphi(\mathbf{p}_2) \varphi^*(\mathbf{p}_1) f(\tau) \quad (54)$$

where $f(\tau)$ is some function of τ , and the form factor $\varphi(\mathbf{p})$ has the following asymptotic behaviour for $|\mathbf{p}| \rightarrow \infty$:

$$\varphi(\mathbf{p}) \sim \frac{c_1}{|\mathbf{p}|^\alpha} \quad (|\mathbf{p}| \rightarrow \infty). \quad (55)$$

In this case, the problem can be easily solved by using equation (44). Representing $\langle \mathbf{p}_2 | T(z) | \mathbf{p}_1 \rangle$ in the form

$$\langle \mathbf{p}_2 | T(z) | \mathbf{p}_1 \rangle = \varphi(\mathbf{p}_2) \varphi^*(\mathbf{p}_1) t(z) \quad (56)$$

from (44) and (45), we get the equation

$$\frac{dt(z)}{dz} = -t^2(z) \int d^3k \frac{|\varphi(\mathbf{k})|^2}{(z - E_k)^2} \quad (57)$$

with the asymptotic condition

$$t(z) \xrightarrow{|z| \rightarrow -\infty} f_1(z) + o(|z|^{-\beta}) \quad (58)$$

where

$$f_1(z) = i \int_0^\infty d\tau \exp(iz\tau) f(\tau) \quad (59)$$

and $E_k = \frac{k^2}{2\mu}$. The solution of equation (57) with the initial condition $t(a) = g_a$, where $a \in (-\infty, 0)$, is

$$t(z) = g_a \left(1 + (z - a) g_a \int d^3k \frac{|\varphi(\mathbf{k})|^2}{(z - E_k)(a - E_k)} \right)^{-1}. \quad (60)$$

In the case $\alpha > \frac{1}{2}$, the function $t(z)$ tends to a constant as $z \rightarrow -\infty$

$$t(z) \xrightarrow{z \rightarrow -\infty} \lambda \quad (61)$$

where

$$\lambda = g_a \left(1 + g_a \int d^3k \frac{|\varphi(\mathbf{k})|^2}{a - E_k} \right)^{-1}. \quad (62)$$

Thus, in this case the function $f_1(z)$ must tend to λ as $z \rightarrow -\infty$. From this and (59) it follows that the only possible form of the function $f(\tau)$ is

$$f(\tau) = -2i\lambda\delta(\tau) + f'(\tau) \quad (63)$$

where the function $f'(\tau)$ has no such a singularity at the point $\tau = 0$ as the delta function. In this case the generalized interaction operator $H_{int}^{(s)}(\tau)$ has the form (48) and hence the dynamics generated by this operator is equivalent to the dynamics governed by the Schrödinger equation with the separable potential

$$\langle \mathbf{p}_2 | H_I | \mathbf{p}_1 \rangle = \lambda \varphi(\mathbf{p}_2) \varphi^*(\mathbf{p}_1). \quad (64)$$

In particular, in this case from (60) and (61) we easily obtain the well known expression for the T matrix in the separable-potential model

$$\langle \mathbf{p}_2 | T(z) | \mathbf{p}_1 \rangle = \lambda \varphi(\mathbf{p}_2) \varphi^*(\mathbf{p}_1) \left(1 - \lambda \int d^3k \frac{|\varphi(\mathbf{k})|^2}{z - E_k} \right)^{-1}. \quad (65)$$

Ordinary quantum mechanics does not permit the extension of the above model to the case $\alpha \leq \frac{1}{2}$. Indeed, in the case of such a large-momentum behaviour of the form factors $\varphi(\mathbf{p})$, substituting the interaction Hamiltonian given by (64) into (51) leads to the ultraviolet divergences, and the integral in (65) is not convergent. We now show that our formalism allows one to extend this model to the case $0 < \alpha < \frac{1}{2}$. Let us determine the class of the functions $f_1(z)$ and correspondingly the value of β for which equation (57) has a unique solution having the asymptotic behaviour (58). In the case $\alpha < \frac{1}{2}$, the function $t(z)$ given by (60) has the following behaviour for $z \rightarrow -\infty$:

$$t(z) \xrightarrow{z \rightarrow -\infty} b_1 |z|^{\alpha-1/2} + b_2 |z|^{2\alpha-1} + o(|z|^{2\alpha-1}) \quad (66)$$

where $b_1 = -\frac{1}{2} \cos(\alpha\pi) \pi^{-2} c_1^{-2} (2\mu)^{\alpha-3/2}$ and $b_2 = b_1 |a|^{1/2-\alpha} - b_1^2 g_a^{-1}$. Here and below we restrict ourselves to the case when $\varphi(\mathbf{p}) = c_1 |\mathbf{p}|^{-\alpha}$. The parameter b_1 does not depend on g_a . This means that all solutions of equation (57) have the same leading term in (66), and only the second term distinguishes the different solutions of this equation. Thus, in order to obtain a unique solution of equation (57) we must specify the first two terms in the asymptotic behaviour of $t(z)$ for $z \rightarrow -\infty$. From this it follows that the functions $f_1(z)$ must be of the form

$$f_1(z) = b_1 |z|^{\alpha-1/2} + b_2 |z|^{2\alpha-1} \quad (67)$$

and $\beta = 2\alpha - 1$. Correspondingly the functions $f(\tau)$ must be of the form

$$f(\tau) = a_1 \tau^{-\alpha-1/2} + a_2 \tau^{-2\alpha} \quad (68)$$

with $a_1 = -ib_1\Gamma^{-1}(1-2\alpha)\exp[i(-\alpha/2+\frac{1}{4})\pi]$, and $a_2 = -b_2\Gamma^{-1}(1-2\alpha)\exp(-i\alpha\pi)$, where $\Gamma(z)$ is the gamma-function. This means that in the case $\alpha < \frac{1}{2}$, the generalized interaction operator must be of the form

$$\langle p_2 | H_{int}^{(s)}(\tau) | p_1 \rangle = a_1 \varphi(p_2) \varphi^*(p_1) \tau^{-\alpha-1/2} + a_2 \varphi(p_2) \varphi^*(p_1) \tau^{-2\alpha}. \quad (69)$$

Using (56) and (60), for $\langle p_2 | T(z) | p_1 \rangle$, we get

$$\langle p_2 | T(z) | p_1 \rangle = N(z) \varphi(p_2) \varphi^*(p_1) \quad (70)$$

with

$$N(z) = g_a \left(1 + (z-a)g_a \int d^3k \frac{|\varphi(k)|^2}{(z-E_k)(a-E_k)} \right)^{-1} \quad (71)$$

where

$$g_a = \frac{b_1^2}{b_1|a|^{1/2-\alpha} + a_2\Gamma(1-2\alpha)\exp(-i\alpha\pi)}.$$

It can be easily checked that $N(z)$ given by (71) does not depend on the choice of the parameter a , and we can let $a \rightarrow -\infty$. Taking into account that

$$g_a \xrightarrow{a \rightarrow -\infty} b_1|a|^{\alpha-\frac{1}{2}} + b_2|a|^{2\alpha-1} + o(|a|^{2\alpha-1})$$

and letting $a \rightarrow -\infty$ in (71), we get

$$N(z) = \frac{b_1^2}{b_1(-z)^{1/2-\alpha} - b_2}. \quad (72)$$

In order that the operator $T(z)$ given by (70) and (72) satisfy equations (33) and (34) the parameter b_2 must be real. The same formula for $T(z)$ can be obtained by using equation (37). It can be shown that if $H_{int}(t_2, t_1)$ is of the form (69), then equation (37) has a unique solution having the behaviour (43) with $\epsilon = -2\alpha$. In the case when $\varphi(p) = c_1|p|^{-\alpha}$, this solution written for $\tilde{T}(\tau)$ is

$$\langle p_2 | \tilde{T}(\tau) | p_1 \rangle = \sum_{n=1}^{\infty} a_n \tau^{n(1/2-\alpha)-1} \varphi(p_2) \varphi^*(p_1)$$

where

$$a_n = a_2^{n-1} a_1^{2-n} \Gamma^{n-1}(1-2\alpha) \Gamma^{2-n}(\frac{1}{2}-\alpha) \Gamma^{-1}(n/2-n\alpha).$$

Substituting this series into (18) we get $\langle p_2 | T(z) | p_1 \rangle$ which can be represented in the form (70). By using (10), (26) and (70), we can construct the evolution operator

$$\begin{aligned} \langle p_2 | U(t, t_0) | p_1 \rangle &= \langle p_2 | p_1 \rangle - \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \\ &\times \frac{\exp[-i(z-E_{p_2})t] \exp[i(z-E_{p_1})t_0]}{(z-E_{p_2})(z-E_{p_1})} N(z) \varphi(p_2) \varphi^*(p_1) \end{aligned} \quad (73)$$

where $z = x + iy$, and $y > 0$. Since $\langle p_2 | T(z) | p_1 \rangle$ given by (70) satisfies equations (33)–(35), the evolution operator $U(t, t_0)$ defined by (12) is a unitary operator satisfying the composition law (4).

We have stated the correspondence between the form of the generalized interaction operator and the large-momentum behaviour of the form factor $\varphi(p)$. In the case $\alpha > \frac{1}{2}$, the operator $H_{int}^{(s)}(\tau)$ would necessarily have the form (48). In this case the fundamental interaction is instantaneous. In the case $0 < \alpha < \frac{1}{2}$ (the restriction $\alpha > 0$ is necessary for the integral in (60) to be convergent), the only possible form of $H_{int}^{(s)}(\tau)$ is (69), and hence the

interaction generating the dynamics of the system is nonlocal in time. In appendix D it is shown that in this case there are normalized vectors in the Hilbert space for which $\langle \psi_2 | U(t, 0) | \psi_1 \rangle$ are not continuous at $t = 0$. These vectors represent the states with infinite energy which are not physically realizable and hence the condition (7) is not violated. Nevertheless, the evolution operator $V(t) = U_s(t, 0)$ is not continuous and hence the group of these operators has no infinitesimal generator in this case. From this it follows that in this case the time evolution of a state vector is not governed by the Schrödinger equation. The cause of this discontinuity is quite obvious. Indeed, from the point of view of the states with infinite energy any time interval δt is infinite and hence the corresponding matrix elements of the evolution operator $U(\delta t, 0)$ must be independent of δt , i.e. must be constant. As is shown in appendix D, in the case $\alpha > \frac{1}{2}$ such matrix elements of $R(t, 0)$ being independent of t are zero, and the discontinuity problem does not appear. This problem appears in the case $\alpha < \frac{1}{2}$, i.e. in the case when the matrix elements $\langle \mathbf{k}_2 | U(t_2, t_1) | \mathbf{k}_1 \rangle$ as functions of \mathbf{k}_1 and \mathbf{k}_2 do not meet the requirements of ordinary quantum mechanics. Thus, the cause of the above-mentioned lack of continuity of the evolution operator is the ‘bad’ large-momentum behaviour of the form factors $\varphi(\mathbf{p})$.

Let us now show that the evolution operator defined by (71) and (73) satisfies condition (7). Using (71) and (73), for the matrix element of the evolution operator in the Schrödinger picture $V(t)$, we can write

$$\langle \psi_2 | V(t) | \psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle - \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) \frac{b_1^2 J(z)}{b_1(-z)^{1/2-\alpha} - b_2} \quad (74)$$

with

$$J(z) = \int d^3 k_1 \int d^3 k_2 \frac{|c_1|^2 \psi_1(\mathbf{k}_1) \psi_2^*(\mathbf{k}_2)}{|\mathbf{k}_1|^\alpha |\mathbf{k}_2|^\alpha (z - E_{k_1})(z - E_{k_2})} \quad (75)$$

where $z = x + iy$, $y > 0$, and $\psi_i(\mathbf{k}) = \langle \mathbf{k} | \psi_i \rangle$, $i = 1, 2$. By using the fact that the functions $\psi_1(\mathbf{k})$ and $\psi_2(\mathbf{k})$ must be square-integrable, it can be shown that, if the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ do not represent the above states with infinite energy, then the function $J(z)$ given by (75) is estimated as

$$J(z) \leq C(1 + |z|)^{-1/2-\alpha-\epsilon}$$

where $\epsilon > 0$. From this and the properties of Fourier integrals it follows that for the physically realizable states the matrix elements $\langle \psi_2 | V(t) | \psi_1 \rangle$ given by (74) are continuous functions of time for $-\infty < t < \infty$ and hence the evolution operator devined by (71) and (73) satisfies condition (7).

7. Summary and discussions

We have constructed a formalism based on the above assumptions of the Feynman formulation and canonical quantum theory. By using assumption (iii) in the spirit of the Feynman formalism, within the formalism the probability amplitude $\langle \psi_2 | U(t, t_0) | \psi_1 \rangle$ is represented as a sum of contributions from the evolution processes associated with completely specified instants of the beginning and end of the interaction in a quantum system, and $\langle \psi_2 | \tilde{S}(t_2, t_1) | \psi_1 \rangle$ represents the contribution from the process in the case where the interaction begins at time t_1 and ends at time t_2 . We have shown that this representation and the requirement of unitarity for the evolution operator lead to equation (37) for the operator $\tilde{S}(t_2, t_1)$ and equations (33)–(35) for the operator $T(z)$ defined by (18). Equation (37) allows one to obtain the operator $\tilde{S}(t_2, t_1)$ if the generalized interaction operator $H_{int}(t_2, t_1)$ which determines the behaviour of $\tilde{S}(t_2, t_1)$ in the limit $t_2 \rightarrow t_1$ is specified. It has been shown that the evolution operator constructed in this way is a unitary operator with the group property (4). In the case when $H_{int}^{(s)}(\tau)$ is of the form (48), the

Schrödinger equation has been shown to result from equation (37) which can be regarded as an equation of motion. In this case the fundamental interaction in the system is instantaneous and the dynamics governed by this equation of motion is equivalent to the Hamiltonian dynamics. At the same time, our formalism permits the generalization to the case when the fundamental interaction in a quantum system being described by the operator $H_{int}^{(s)}(\tau)$ is nonlocal in time. In this case the dynamics is not equivalent to the dynamics governed by the Schrödinger equation. This is not at variance with Stone's theorem, since, as it has been shown on the exactly solvable model, the evolution operator is not strongly continuous in this case. It only satisfies the more general continuity condition (7). We have stated correspondence between the large-momentum behaviour of the matrix elements of the evolution operator and the form of the generalized interaction operator $H_{int}^{(s)}(\tau)$: if this behaviour satisfies the requirements of conventional quantum theory, then $H_{int}^{(s)}(\tau)$ must necessarily be of the form (48), and if this large-momentum behaviour does not meet the above requirements, then $H_{int}^{(s)}(\tau)$ must be of the form corresponding to the case when the interaction generating the dynamics of a quantum system is nonlocal in time.

The above gives reason to hope that the proposed formalism may open new possibilities for solving the problem of the ultraviolet divergences in quantum field theory. In this connection note the following. Since locality is the main cause of the ultraviolet divergences, it seems natural to resolve the problem by the introduction of a nonlocal form factor into the interaction Hamiltonian density. For example, in QED a nonlocal form factor could be introduced by specifying the interaction Hamiltonian density in the form

$$\mathcal{H}_I(x) = \int d^4x_1 \int d^4x_2 F(x - x_1, x - x_2) j_\mu(x_1) A^\mu(x_2)$$

where $F(x - x_1, x - x_2)$ is a covariant form factor, $j_\mu(x)$ is the current density operator, and $A_\mu(x)$ is the electromagnetic field potential. However, it turned out that the nonlocal form factor cannot be introduced in such a way without losing covariance of the theory. Indeed, for the theory to be relativistically invariant, $\mathcal{H}_I(x)$ must satisfy the condition

$$\mathcal{H}_I(x_1)\mathcal{H}_I(x_2) = \mathcal{H}_I(x_2)\mathcal{H}_I(x_1) \quad (x_2 - x_1)^2 \langle 0.$$

But, as is well known, the introduction of the nonlocal form factor leads to a violation of this condition. The reason for these difficulties is quite obvious. Due to the delta function $\delta(\tau)$ the generalized interaction operator given by (48) describes an instantaneous interaction. In nonrelativistic quantum mechanics processes of instantaneous interaction may be nonlocal in space. However, in relativistic quantum theory a local-in-time process must be also local in space. Thus, in the case when the generalized interaction operator $H_{int}^{(s)}(\tau)$ has the form (48), the theory is local in nature, and the introduction of a nonlocal form factor leads to the intrinsic inconsistency of the theory. On the other hand, our formalism permits a generalization to the class of the operators $H_{int}^{(s)}(\tau)$ that are not of the form (48). In this case the duration time $\tau = t_2 - t_1$ of the fundamental interaction is not zero, and the description of the dynamics of the system becomes fundamentally nonlocal. For example, in QED the operator $H_{int}(t_2, t_1)$ may be specified in the form

$$H_{int}(t_2, t_1) = \int d^3x_1 \int d^3x_2 \mathcal{H}_{int}(x_2, t_2, x_1, t_1)$$

with

$$\mathcal{H}_{int}(x_2, x_1) = \int d^4y_1 \int d^4y_2 F(x_2 - x_1, x_2 - y_1, x_2 - y_2) j_\mu(y_1) A^\mu(y_2).$$

Here $F(x_2 - x_1, x_2 - y_1, x_2 - y_2)$ is a relativistically invariant form factor. The matrix elements $\langle \psi_2 | \mathcal{H}_{int}(x_2, x_1) | \psi_1 \rangle$ may be interpreted as amplitudes describing processes in which

interaction begins at a point x_1 of space-time and ends at a point x_2 , and the presence of the nonlocal form factor in the expression of $\mathcal{H}_{int}(x_2, x_1)$ seems to be natural. One may hope that the form factor $F(x_2 - x_1, x_2 - y_1, x_2 - y_2)$ can be chosen in such a way that no divergence difficulties arise in describing the time evolution of systems in QED, and, at the same time, solution of equation (37) lead to the ordinary renormalized expression for the S matrix.

Let us now show that the proposed formalism may open new possibilities for describing the low-energy nucleon dynamics. For this purpose, return to the model considered in section 6. In the connection with the fact that this model is an extension of the ordinary separable-potential model to the case when the separable interaction is nonlocal in both space and time, note that separable potentials are widely used in nuclear physics (see, for example, [25–30]). One may hope that within this extended model a better agreement with experiment can be achieved.

In the case when the operator $H_{int}(t_2, t_1)$ has such a form that the interaction generating the dynamics of a system is nonlocal in time, the effective interaction operator $B(z)$ is energy dependent. As mentioned above, the energy dependence of the effective operators of the hadron–hadron interactions is associated with the quark and gluon degrees of freedom which are not included explicitly in the description of low-energy hadron interaction. It should be noted that, in general, the energy-dependent operator of hadron–hadron interaction cannot also be assumed to be an optical potential, since due to the quark confinement a hadron system can be considered as a closed system, and for conservation of probability one need not to include explicitly in the description the channels associated with the quark and gluon degrees of freedom. A remarkable feature of the proposed formalism is that the operator of interaction may be energy dependent even in the case of closed systems when the time evolution is unitary. At the same time, the quark and gluon degrees of freedom may manifest themselves in the dependence upon the time duration $\tau = t_2 - t_1$ of the interaction. In this case the generalized interaction operator may be interpreted as an operator whose matrix elements $\langle \psi_2 | H_{int}(t_2, t_1) | \psi_1 \rangle$ describe transitions between hadron states, and t_1 and t_2 are the times between which the quark and gluon degrees of freedom come into play. One may hope that it will become possible to construct such operators by using QCD-inspired quark models.

Taking into account the quark and gluon degrees of freedom is especially important for describing the short-range NN interaction. In this connection note that in the most realistic NN potentials theoretical parts are supplemented with phenomenological short-range parts. Our formalism makes it possible to describe these phenomenological parts in terms of the operator $H_{int}(t_2, t_1)$. This may open new possibilities for reducing the discrepancy that still exists between theory and experiment.

Appendix A

Represent (24) in the form

$$\langle n_2 | R(t, t_0) | n_1 \rangle = \lim_{a \rightarrow \infty} \langle n_2 | R_a(t, t_0) | n_1 \rangle$$

where

$$\begin{aligned} \langle n_2 | R_a(t, t_0) | n_1 \rangle = & -\frac{a}{2\pi} \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \int_{-\infty}^{\infty} dx \frac{\langle n_2 | T(z) | n_1 \rangle}{z + a} \\ & \times \exp(iE_{n_2}t_2) \exp[-iz(t_2 - t_1)] \exp(-iE_{n_1}t_1) \end{aligned}$$

and a is real. Integrating over t_1 and t_2 (here the integrations over t_1 , t_2 and x_1 may be reversed in order), we get

$$\langle n_2 | R_a(t, t_0) | n_1 \rangle = \frac{a}{2\pi} \int_{-\infty}^{\infty} dx \frac{\langle n_2 | T(z) | n_1 \rangle}{(z - E_{n_2})(z - E_{n_1})(z + a)}$$

$$\times \exp[-i(z - E_{n_2})t] \exp[i(z - E_{n_1})t_0] + \langle n_2 | B(t, t_0) | n_1 \rangle$$

where

$$\langle n_2 | B(t, t_0) | n_1 \rangle = \int_{-\infty}^{\infty} dx \langle n_2 | F(z, t, t_0) | n_1 \rangle$$

with

$$\begin{aligned} \langle n_2 | F(z, t, t_0) | n_1 \rangle &= -\frac{a}{2\pi} \frac{\langle n_2 | T(z) | n_1 \rangle}{(z + a)} \\ &\times \left[\frac{\exp[i(E_{n_2} - E_{n_1})t_0]}{(z - E_{n_2})(z - E_{n_1})} - \frac{\exp[i(E_{n_2} - E_{n_1})t] - \exp[i(E_{n_2} - E_{n_1})t_0]}{(z - E_{n_2})(E_{n_2} - E_{n_1})} \right]. \end{aligned}$$

Taking into account the above analytic properties of $\langle n_2 | T(z) | n_1 \rangle$ and the asymptotic condition (25), we get

$$\int_{-\infty}^{\infty} dx \langle n_2 | F(z, t, t_0) | n_1 \rangle = \oint_{c_1} dz \langle n_2 | F(z, t, t_0) | n_1 \rangle = 0$$

where $z = x + iy$, $y > 0$, and the contour of integration c_1 consists of the axis $y = \text{constant}$ and the upper half of the infinite circle. Thus, for $\langle n_2 | R(t, t_0) | n_1 \rangle$ we have

$$\begin{aligned} \langle n_2 | R(t, t_0) | n_1 \rangle &= \lim_{a \rightarrow \infty} \left(\frac{a}{2\pi} \int_{-\infty}^{\infty} dx \langle n_2 | T(z) | n_1 \rangle \frac{\exp[-i(z - E_{n_2})t] \exp[i(z - E_{n_1})t_0]}{(z - E_{n_2})(z - E_{n_1})(z + a)} \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\exp[-i(z - E_{n_2})t] \exp[i(z - E_{n_1})t_0]}{(z - E_{n_2})(z - E_{n_1})} \langle n_2 | T(z) | n_1 \rangle. \end{aligned}$$

Appendix B

Let us show that the evolution operators given by (10) and (26) satisfies the composition law (4), provided equation (35) is valid. The right-hand side of (4) can be rewritten as follows:

$$U_r(t, t', t_0) \equiv U(t, t')U(t', t_0) = \mathbf{1} + iR(t, t') + iR(t', t_0) - R(t, t')R(t', t_0).$$

Let $t \geq t' \geq t_0$. By using (26), for $\langle n_2 | U_r(t, t', t_0) | n_1 \rangle$, we get

$$\langle n_2 | U_r(t, t', t_0) | n_1 \rangle = \langle n_2 | n_1 \rangle + i\langle n_2 | R(t, t') | n_1 \rangle + i\langle n_2 | R(t', t_0) | n_1 \rangle + \langle n_2 | D(t, t', t_0) | n_1 \rangle$$

where

$$\begin{aligned} \langle n_2 | D(t, t', t_0) | n_1 \rangle &= -\frac{1}{(2\pi)^2} \sum_n \int dx_1 \int dx_2 \exp[-i(z_2 - E_{n_2})t] \\ &\times \exp[i(z_1 - E_{n_1})t_0] \exp[i(z_2 - z_1)t'] \\ &\times \frac{\langle n_2 | T(z_2) | n \rangle \langle n | T(z_1) | n_1 \rangle}{(z_2 - E_{n_2})(z_2 - E_n)(z_1 - E_n)(z_1 - E_{n_1})} \end{aligned}$$

with $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $y_2 > y_1$. Assuming that $T(z)$ satisfies equation (35), this equation can be written in the form

$$\begin{aligned} \langle n_2 | D(t, t', t_0) | n_1 \rangle &= -\frac{1}{(2\pi)^2} \int dx_1 \int dx_2 \\ &\times \frac{\exp[-i(z_2 - E_{n_2})t] \exp[i(z_1 - E_{n_1})t_0] \exp[i(z_2 - z_1)t']}{(z_2 - E_{n_2})(z_1 - E_{n_1})} \\ &\times \left[\frac{\langle n_2 | T(z_1) | n_1 \rangle}{z_2 - z_1} - \frac{\langle n_2 | T(z_2) | n_1 \rangle}{z_2 - z_1} \right] = \frac{i}{2\pi} \int dx_1 \\ &\times \{ \exp[-i(z_1 - E_{n_2})t] \exp[i(z_1 - E_{n_1})t_0] - \exp[-i(z_1 - E_{n_2})t] \} \end{aligned}$$

$$\begin{aligned}
& \times \exp[i(z_1 - E_{n_1})t'] \} \{ (z_1 - E_{n_2})(z_1 - E_{n_1}) \}^{-1} \\
& \times \langle n_2 | T(z_1) | n_1 \rangle - \frac{i}{2\pi} \int dx_2 \frac{\exp[-i(z_2 - E_{n_2})t'] \exp[i(z_2 - E_{n_1})t_0]}{(z_2 - E_{n_2})(z_2 - E_{n_1})} \\
& \times \langle n_2 | T(z_2) | n_1 \rangle = i(\langle n_2 | R(t, t_0) | n_1 \rangle - \langle n_2 | R(t, t') | n_1 \rangle - \langle n_2 | R(t', t_0) | n_1 \rangle).
\end{aligned}$$

From this equation it follows that equation (4) is satisfied for $t \geq t' \geq t_0$. By using (13), it is easy to show then that the evolution operator satisfies the composition law (4) for any t, t' and t_0 .

Appendix C

Let us now directly derive an equation for the operator $\tilde{S}(t_2, t_1)$ starting from equation (35). From this equation, using the definition of $\langle n_2 | T(z) | n_1 \rangle$ given by (27), one can find

$$\begin{aligned}
& \int_0^\infty d\tau \exp(iz_1\tau) \tau (1 - \exp[i(z_2 - z_1)\tau]) \langle n_2 | \tilde{T}(\tau) | n_1 \rangle \\
& = \sum_n \int_0^\infty d\tau \exp(iz_1\tau) \int_0^\tau d\tau_2 \int_0^{\tau-\tau_2} d\tau_1 (\tau - \tau_1 - \tau_2) \\
& \quad \times \exp[-iE_n(\tau - \tau_1 - \tau_2)] \langle n_2 | \tilde{T}(\tau_2) | n \rangle \langle n | \tilde{T}(\tau_1) | n_1 \rangle.
\end{aligned}$$

For this relation to be valid for any z_1 and z_2 the matrix elements $\langle n_2 | \tilde{T}(\tau) | n_1 \rangle$ must satisfy the following equation:

$$\begin{aligned}
& (1 - \exp[i(z_2 - z_1)\tau]) \tau \langle n_2 | \tilde{T}(\tau) | n_1 \rangle \\
& = \sum_n \int_0^\tau d\tau_2 \int_0^{\tau-\tau_2} d\tau_1 (\tau - \tau_1 - \tau_2) \exp[-iE_n(\tau - \tau_1 - \tau_2)] \\
& \quad \times \langle n_2 | \tilde{T}(\tau_2) | n \rangle \langle n | \tilde{T}(\tau_1) | n_1 \rangle.
\end{aligned}$$

From this equation, by using (15), one can get equation (36) for the operator $\tilde{S}(t_2, t_1)$. Equation (36) has been rewritten in the form

$$\begin{aligned}
& \left[\sum_{n=1}^\infty \frac{(ia)^n}{n!} (t_2 - t_1)^n \right] \tilde{S}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 \\
& \quad \times \left[\sum_{m=0}^\infty \frac{(ia)^m}{m!} (t_2 - t_4)^m \right] \left[\sum_{k=1}^\infty \frac{(ia)^k}{k!} (t_4 - t_3)^k \right] \tilde{S}(t_2, t_4) \tilde{S}(t_3, t_1).
\end{aligned}$$

For this relation to be valid for every a , each term of the expansion in powers of a on the left-hand side of this equation must be equal to the term of the same order on its right-hand side. In the first order, for example, we have

$$a(t_2 - t_1) \tilde{S}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3) \tilde{S}(t_2, t_4) \tilde{S}(t_3, t_1).$$

From this it follows that $\tilde{S}(t_2, t_1)$ must satisfy equation (37).

Appendix D

Let us show that in the case $\alpha < \frac{1}{2}$ there are states in the Hilbert space for which $\langle \psi_2 | U(t, 0) | \psi_1 \rangle$ are not continuous. Using (10), (12) and (15), for $\langle \psi_2 | R(t, 0) | \psi_1 \rangle$, we can write

$$\begin{aligned}
& \langle \psi_2 | R(t, 0) | \psi_1 \rangle = i \int_0^t dt_2 \int_0^{t_2} dt_1 \int d^3k_1 \int d^3k_2 \exp[i(E_{k_2} - E_{k_1})t_2] \\
& \quad \times \exp[iE_{k_1}(t_2 - t_1)] \psi_2^*(\mathbf{k}_2) \psi_1(\mathbf{k}_1) \langle \mathbf{k}_2 | \tilde{T}(t_2 - t_1) | \mathbf{k}_1 \rangle
\end{aligned}$$

where $\psi_i(\mathbf{k}) = \langle \mathbf{k} | \psi_i \rangle$, $i = 1, 2$. Consider the vectors $|\psi_v\rangle$ for which

$$\langle \mathbf{k} | \psi_v \rangle = \frac{vd}{k(k - v - i\nu\Gamma_0)}$$

where $d = \Gamma_0^{1/2}(2\pi)^{-1/2}[\pi/2 + \arctg(\frac{1}{\Gamma_0})^{-1}]$, $k = |\mathbf{k}|$, and Γ_0 is a real constant. It is easy to verify that these vectors are normalized.

Let us now consider the case $\alpha > \frac{1}{2}$. Taking into account (56) and (63), for $\langle \psi_v | R(t, 0) | \psi_v \rangle$ in the limit $\nu \rightarrow \infty$, we have

$$\begin{aligned} \langle \psi_v | R(t, 0) | \psi_v \rangle &= \nu^{1-2\alpha} g d^2 |c_1|^2 \int_0^\theta d\theta_1 \int d^3 q_1 \int d^3 q_2 \\ &\times \frac{\exp[i(E_{q_2} - E_{q_1})\theta_1]}{(q_1 - 1 + i\Gamma_0)(q_2 - 1 - i\Gamma_0)q_1^\alpha q_2^\alpha} + o(\nu^{1-2\alpha}) \end{aligned}$$

where $q_i = k_i \nu^{-1}$, $\theta = t \nu^2$, $\theta_1 = t_1 \nu^2$. It is easy to see that $\langle \psi_v | R(t, 0) | \psi_v \rangle$ given by this equation tends to zero as $\nu \rightarrow \infty$. Let us now consider the case $\alpha < \frac{1}{2}$. According to (69), $\langle \mathbf{k}_2 | \tilde{T}(\tau) | \mathbf{k}_1 \rangle$ has the following behaviour near the point $\tau = 0$:

$$\langle \mathbf{k}_2 | \tilde{T}(\tau) | \mathbf{k}_1 \rangle = a_1 \varphi(\mathbf{k}_2) \varphi^*(\mathbf{k}_1) \tau^{-\beta} + o(\tau^{-\beta})$$

with $\beta = \alpha - \frac{1}{2}$. Taking into account this fact, for $\langle \psi_v | R(t, 0) | \psi_v \rangle$ in the limit $\nu \rightarrow \infty$, we get

$$\langle \psi_{k_0} | R(t, 0) | \psi_{k_0} \rangle = \nu^{\alpha-1/2-\beta} \int_0^\theta d\theta_2 \int_0^{\theta_2} d\theta_1 F_2(\theta_1, \theta_2) + o(\nu^{\alpha+\frac{1}{2}-\beta})$$

where

$$F_2(\theta_1, \theta_2) = a_1 d^2 |c_1|^2 \int d^3 q_1 \int d^3 q_2 \frac{\exp[i(E_{q_2} - E_{q_1})\theta_2] \exp[iE_{q_1}(\theta_2 - \theta_1)]}{(q_1 - 1 + i\Gamma_0)(q_2 - 1 - i\Gamma_0)q_1^{1+\alpha} q_2^{1+\alpha} (\theta_2 - \theta_1)^\beta}$$

$q_i = k_i \nu^{-1}$, $\theta = t_i \nu^2$, and $\theta = t \nu^2$. Since $\beta = \alpha - \frac{1}{2}$, it follows from this that

$$\lim_{\nu \rightarrow \infty} \langle \psi_v | R(t, 0) | \psi_v \rangle = C$$

where $C = \int_0^\infty d\theta_2 \int_0^\infty d\theta_1 F_2(\theta_1, \theta_2)$. Thus, in the limit $\nu \rightarrow \infty$ the matrix element $\langle \psi_v | R(t, 0) | \psi_v \rangle$ is independent of t . This means that there are normalized vectors in the Hilbert space for which $\langle \psi_2 | U(t, 0) | \psi_1 \rangle$ does not tend to $\langle \psi_2 | \psi_1 \rangle$ as $t \rightarrow 0$. However, these vectors represent the states with infinite momentum and correspondingly with infinite energy which are not physically realizable and hence condition (7) is not violated.

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